# PARTIAL REGULARITY FOR A LIOUVILLE SYSTEM

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth open set. We prove that the singular set of any extremal solution of the system

$$-\Delta u = \mu e^v, \quad -\Delta v = \lambda e^u \quad \text{in } \Omega,$$

with $u = v = 0$ on $\partial\Omega$, $\mu, \lambda \geq 0$, has Hausdorff dimension at most $n - 10$.

## 1. Introduction

In this article we consider the issue of partial regularity of extremal solutions to the Liouville system

$$\begin{cases} -\Delta u = \mu e^v & \text{in } \Omega, \\ -\Delta v = \lambda e^u & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with $\Omega$ a bounded smooth open subset of $\mathbb{R}^n$, and $\lambda, \mu$ nonnegative parameters.

This system is a generalization of the equation

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\lambda$ denotes a positive parameter. It is well known that there is a maximal parameter $\lambda^* > 0$ for existence of solutions of (2) and for $0 < \lambda < \lambda^*$ there is a minimal solution $u_\lambda$. As $\lambda \to \lambda^*, \lambda < \lambda^*$ the solution $u_\lambda$ converges to the so-called extremal solution, which turns out to be smooth for $n \leq 9$, see [3, 11]. The interested reader may find in the book [7] the developments of the theory for the last six decades, with a particular focus on stable solutions.

Recently it was proved by K. Wang [13] that for $n \geq 10$ the extremal solution of (2) has a singular set of dimension at most $n - 10$. F. Da Lio [5] obtained partial regularity for any weak *stationary* solution in dimension 3 (not necessarily stable). See related results for the Lane-Emden equation in [14, 6].

Here we generalize the results of [13] to the system (1). For this system, M. Montenegro [12] proved the existence of a nonempty open set $\mathcal{U}$ in the quarter plane $\lambda, \mu > 0$ such that for a couple of parameters $(\mu, \lambda)$ in $\mathcal{U}$ there is a smooth *minimal* solution $(u, v)$ and no smooth solution exists if the couple is in the complement of

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$\overline{\mathcal{U}}$. Minimality means $u \le \tilde{u}$ and $v \le \tilde{v}$ in $\Omega$ for any other smooth solution $(\tilde{u}, \tilde{v})$ for the same $(\mu, \lambda)$.

For each slope $m > 0$, $\mathcal{U}$ intersected with the line $\mu = m\lambda$ is a segment $\{(m\lambda, \lambda) : \lambda \in (0, \lambda^*(m))\}$ and at the extremal point $(m\lambda^*(m), \lambda^*(m)) \in \partial\mathcal{U}$ there is a solution, called the extremal solution. It is defined as the limit as $\lambda \uparrow \lambda^*(m)$ of the minimal solution with parameters $(m\lambda, \lambda)$ and it may be singular. In a recent work [8], L. Dupaigne, A. Farina and B. Sirakov proved that the extremal solutions for the Liouville system (1) are smooth if $n \le 9$. C. Cowan [1] had obtained the same conclusion under the restrictions $3 \le n \le 9$ and $\frac{n-2}{8} \le \frac{\mu}{\lambda} \le \frac{8}{n-2}$. In higher dimensions this fails at least in the radial case and for $\lambda = \mu$, where (1) reduces to (2).

Let us recall that en extremal solution $(u, v)$ satisfies (1) in the sense that $u, v \in L^1(\Omega)$, $e^u\, dist(\cdot, \partial\Omega), e^v dist(\cdot, \partial\Omega) \in L^1(\Omega)$, and

$$\int_\Omega u(-\Delta\varphi) = \int_\Omega \mu e^v \varphi, \quad \int_\Omega v(-\Delta\varphi) = \int_\Omega \lambda e^u \varphi,$$

for all $\varphi \in C^2(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.

We define the singular set $\Sigma$ of an extremal solution $(u, v)$ by $x \notin \Sigma$ if there is a neighborhood $W$ of $x$ such that $u$, $v$ are bounded in $W$. By elliptic regularity, $u$, $v$ are then smooth in this neighborhood.

**Theorem 1.1.** *Assume $n \ge 10$ and let $(u, v)$ be an extremal solution of the Liouville system (1) and $\Sigma$ be its singular set. Then the Hausdorff dimension of $\Sigma$ is less or equal than $n - 10$.*

The rest of the article is devoted to the proof of this theorem. We first recall a useful inequality which is valid for stable solutions of the system, obtained in C. Cowan, N. Ghoussoub [2] and L. Dupaigne, A. Farina, B. Sirakov [8]. We then state a comparison result between $u$ and $v$. Next, we perform a Moser iteration scheme to control the growth of some integrals of $e^u$ and $e^v$ on balls. The final step is an adaptation of an argument of K. Wang [13] using an $\varepsilon$-regularity result. The result in this paper is also closely related to the work of L. Dupaigne, M. Ghergu, O. Goubet and G. Warnault [9] on stable solutions of $\Delta^2 u = e^u$ in a bounded domain or entire space.

## 2. Proof of Theorem 1.1

From [12] we know that for $(\mu, \lambda) \in \mathcal{U}$, the associated minimal solution $(u, v)$ of (1), which is smooth, is stable in the sense that there exist $\varphi, \psi : \Omega \to \mathbb{R}$, smooth and positive in $\Omega$, satisfying

$$\begin{cases} -\Delta\varphi - \mu e^v \psi = \eta\varphi & \text{in } \Omega, \\ -\Delta\psi - \lambda e^u \varphi = \eta\psi & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $\eta > 0$. C. Cowan, N. Ghoussoub [2] and independently L. Dupaigne, A. Farina, B. Sirakov [8] have showed that this stability condition implies the following estimate.

**Lemma 2.1.** *Let $(u, v)$ be a smooth stable solution of the system (1). For any $\varphi$ in $H_0^1(\Omega)$*

$$\sqrt{\lambda\mu} \int_\Omega \exp\left(\frac{u+v}{2}\right)\varphi^2 \le \int_\Omega |\nabla\varphi|^2. \tag{3}$$

2.1. **Comparison.** It will be useful later to have the following inequalities between the components of a solution of (1).

**Lemma 2.2.** *Assume $\lambda \geq \mu$. Then for any smooth solution to the Liouville system (1) we have:*

$$u \leq v \leq u + \log\lambda - \log\mu. \tag{4}$$

*Proof.* Introduce $w = v - u - \log\lambda + \log\mu$. Then $w \leq 0$ on $\partial\Omega$. We have $-\Delta w = \lambda e^u - \mu e^v = -\lambda e^u(e^w - 1)$, and then

$$-\Delta w + \lambda e^u\left(\frac{e^w - 1}{w}\right)w = 0.$$

Then due to the maximum principle $w \leq 0$ in $\Omega$. For the first inequality in (4) introduce $\tilde{w} = v - u$. Then $-\Delta\tilde{w} = \lambda e^u - \mu e^v \geq \lambda(e^u - e^v) = -a(x)\tilde{w}$ where $a(x) \geq 0$. Then by the maximum principle $\tilde{w} \geq 0$ in $\Omega$. $\square$

2.2. **Reverse Hölder inequality.** The following estimate is similar to the one obtained in [8] and [9], see also [4] for the scalar case. We assume that $(u, v)$ is a smooth stable solution of (1).

**Lemma 2.3.** *For any $0 < \alpha < 4$ there exists a constant $C = C(n, \alpha, \lambda, \mu)$ such that for any $\varphi \in C_c^\infty(\Omega)$ we have*

$$\begin{aligned} &\|\nabla(\exp(\frac{\alpha u}{2})\varphi)\|^2_{L^2(\Omega)} + \|\nabla(\exp(\frac{\alpha v}{2})\varphi)\|^2_{L^2(\Omega)} \\ &\qquad \leq C\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 + |\varphi\Delta\varphi|^2) + C\int_\Omega e^{\alpha v}(|\nabla\varphi|^2 + |\varphi\Delta\varphi|^2). \end{aligned} \tag{5}$$

**Remark 1.** Although the constant $C$ depends on $\mu, \lambda$ it remains bounded as $(\mu, \lambda)$ approaches any extremal couple on $\partial\mathcal{U}$.

*Proof.* Multiply $-\Delta u = \mu e^v$ by $e^{\alpha u}\varphi^2$ and integrate by parts to obtain

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 = \int_\Omega \nabla u \nabla(e^{\alpha u}\varphi^2) = \frac{4}{\alpha}\int_\Omega \varphi^2|\nabla(e^{\frac{\alpha u}{2}})|^2 + \frac{1}{\alpha}\int_\Omega \nabla(e^{\frac{\alpha u}{2}})\nabla\varphi^2.$$

This reads also

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 = \frac{4}{\alpha}\int_\Omega |\nabla(e^{\frac{\alpha u}{2}}\varphi)|^2 - \frac{2}{\alpha}\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 - \varphi\Delta\varphi).$$

A similar equality is valid replacing respectively $u$ by $v$ and $\mu$ by $\lambda$. Introducing $X = \int_\Omega |\nabla(e^{\frac{\alpha u}{2}}\varphi)|^2$, $Y = \int_\Omega |\nabla(e^{\frac{\alpha v}{2}}\varphi)|^2$, $A = \frac{2}{\alpha}\int_\Omega e^{\alpha u}(|\nabla\varphi|^2 - \varphi\Delta\varphi)$, and $B = \frac{2}{\alpha}\int_\Omega e^{\alpha v}(|\nabla\varphi|^2 - \varphi\Delta\varphi)$, we then have

$$\begin{aligned} \frac{4}{\alpha}X &= \mu\int_\Omega e^{v+\alpha u}\varphi^2 + A, \\ \frac{4}{\alpha}Y &= \lambda\int_\Omega e^{u+\alpha v}\varphi^2 + B. \end{aligned}$$

We combine Hölder's inequality and the stability estimate (3) to obtain

$$\mu\int_\Omega e^{v+\alpha u}\varphi^2 \leq \mu\left(\int_\Omega e^{\frac{u+v}{2}}e^{\alpha u}\varphi^2\right)^{1-\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u+v}{2}}e^{\alpha v}\varphi^2\right)^{\frac{1}{2\alpha}} \leq \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}}X^{1-\frac{1}{2\alpha}}Y^{\frac{1}{2\alpha}}.$$

Analogously, we have the same inequality replacing $u$ by $v$ and $\mu$ by $\lambda$. Hence we obtain

$$\frac{4}{\alpha}X \leq (\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} + A, \tag{6}$$

$$\frac{4}{\alpha}Y \leq (\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} + B. \tag{7}$$

Multiplying these inequalities leads to

$$(\frac{16}{\alpha^2} - 1)XY \leq A(\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} + B(\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} + AB.$$

Set $\delta = (\frac{16}{\alpha^2} - 1)$. This implies that either

$$(\frac{\mu}{\lambda})^{\frac{1}{2}} X^{1-\frac{1}{2\alpha}} Y^{\frac{1}{2\alpha}} \leq \frac{A}{\delta}(1 + \sqrt{1+\delta}), \tag{8}$$

or

$$(\frac{\lambda}{\mu})^{\frac{1}{2}} X^{\frac{1}{2\alpha}} Y^{1-\frac{1}{2\alpha}} \leq \frac{B}{\delta}(1 + \sqrt{1+\delta}) \tag{9}$$

hold. Assuming that (8) is true and combining with (6) we get $X \leq CA$. Using Young's inequality in (7) we obtain $Y \leq C(A+B)$ so that $X+Y \leq C(A+B)$ holds, which is (5). Assuming the validity of (9) we obtain the same conclusion. □

A consequence of the previous lemma is the following.

**Lemma 2.4.** *Set $2^* = \frac{2n}{n-2}$. For any $0 < \alpha < \beta < 2(2^*)$, if $B_{2r}(x) \subset \Omega$ we have*

$$\left( r^{-n} \int_{B_r(x)} (e^{\beta u} + e^{\beta v}) \right)^{\alpha/\beta} \leq C r^{-n} \int_{B_{2r}(x)} e^{\alpha u} + e^{\alpha v} \tag{10}$$

*Proof.* Follows from repeated applications of Lemma 2.3, using Sobolev's embedding and Hölder's inequality. □

**Remark 2.** Lemmas 2.3 and 2.4 are independent of the boundary conditions of $u$ and $v$, and do not use the comparison of $u$ to $v$ of Lemma 2.2.

2.3. **Integrability of solutions.**

**Lemma 2.5.** *Assume $(u, v)$ is a stable smooth solution of (1) with parameter $(\mu, \lambda)$ of the form $\mu = m\lambda$ for some fixed $m > 0$. For $1 \leq \alpha < 5$ there is $C$ independent of $\lambda$ such that*

$$\int_\Omega e^{\alpha u} + e^{\alpha v} \leq C.$$

We note that $C$ in general depends on the slope $m$. In this lemma we need the inequalities between $u$ and $v$ of Lemma 2.2. For the proof, we refer to [8] where the following was proved.

**Lemma 2.6.** *Assume $\lambda \geq \mu$. If $(u, v)$ is a stable smooth solution of (1) with parameter $(\mu, \lambda)$ of the form $\mu = m\lambda$ for some fixed $m > 0$, then for $1 \leq \alpha < 5$ there is $C$ independent of $\lambda$ such that*

$$\int_\Omega e^{\alpha u} \leq C.$$

Lemma 2.5 follows from Lemmas 2.6 and 2.2 in the case $\lambda \geq \mu$. By a symmetric argument we obtain the same conclusion if $\lambda \leq \mu$.

2.4. **$\varepsilon$-regularity.** A crucial step is the following $\varepsilon$-regularity result, whose version for stable solutions in the scalar case is due to K. Wang [13], see also [9] for a biharmonic equation with exponential nonlinearity.

**Lemma 2.7.** *Let $(u, v)$ be an extremal solution of (1). Then there is $\varepsilon_2 > 0$ such that if for some $r_0 > 0$ with $B_{r_0}(x) \subset \Omega$ one has*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon_2$$

*then there is a neighborhood of $x$ such that $u$, $v$ are smooth in this neighborhood.*

For the proof we need the following key step, which is adapted from [13] in the scalar case.

**Lemma 2.8.** *There exists $\varepsilon_0 > 0$ and $\theta > 0$ depending only on $n$ such that for any $0 < \varepsilon \leq \varepsilon_0$, if $(u, v)$ is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \leq \varepsilon \tag{11}$$

*then*

$$(\theta r_0)^{2-n} \int_{B_{\theta r_0}(x)} (e^u + e^v) \leq \varepsilon. \tag{12}$$

*Proof.* Let us assume that $x = 0$ by shifting coordinates. We rescale the functions by setting

$$\tilde{u}(x) = u(r_0 x) + 2\log(r_0), \quad \tilde{v}(x) = v(r_0 x) + 2\log(r_0), \tag{13}$$

and note that the new functions (where the $\tilde{}$ in the notation will be dropped) satisfy

$$-\Delta u = \mu e^v, \quad -\Delta v = \lambda e^u, \quad \text{in } B_1(0).$$

Let us decompose $u = u_1 + u_2$, $v = v_1 + v_2$ where

$$\begin{aligned} \Delta u_1 &= 0 \quad \text{in } B_{1/2}(0), & u_1 &= u \quad \text{on } \partial B_{1/2}(0), \\ -\Delta u_2 &= \mu e^v \quad \text{in } B_{1/2}(0), & u_2 &= 0 \quad \text{on } \partial B_{1/2}(0), \\ \Delta v_1 &= 0 \quad \text{in } B_{1/2}(0), & v_1 &= v \quad \text{on } \partial B_{1/2}(0), \\ -\Delta v_2 &= \lambda e^u \quad \text{in } B_{1/2}(0), & v_2 &= 0 \quad \text{on } \partial B_{1/2}(0). \end{aligned}$$

Let $\gamma > 0$, $0 < \theta < 1/4$ to be fixed later on and $\varepsilon > 0$. Let us estimate

$$\theta^{2-n} \int_{B_\theta(0)} e^u = \theta^{2-n} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1 + u_2} + \theta^{2-n} \int_{B_\theta(0) \cap [u_2 > \varepsilon^\gamma]} e^u. \tag{14}$$

For the first term we proceed by noting that $e^{u_1}$ is subharmonic in $B_{1/2}(0)$ and $u_2 \geq 0$, so

$$\begin{aligned} \theta^{2-n} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1 + u_2} &\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_\theta(0) \cap [u_2 \leq \varepsilon^\gamma]} e^{u_1} \\ &\leq \theta^{2-n} e^{\varepsilon^\gamma} \int_{B_\theta(0)} e^{u_1} \\ &\leq C \theta^2 e^{\varepsilon^\gamma} \int_{B_{1/2}(0)} e^{u_1} \\ &\leq C \theta^2 e^{\varepsilon^\gamma} \int_{B_{1/2}(0)} e^u \leq C \theta^2 e^{\varepsilon^\gamma} \varepsilon, \end{aligned} \tag{15}$$

where we have used (11). For the second term in (14) we have

$$\begin{aligned}\theta^{2-n}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} e^u &\le \theta^{2-n}\varepsilon^{-\gamma}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} u_2 e^u\\ &\le \theta^{2-n}\varepsilon^{-\gamma}\int_{B_{1/2}(0)} u_2 e^u\\ &\le \theta^{2-n}\varepsilon^{-\gamma}\|u_2\|_{L^2(B_{1/2}(0))}\|e^u\|_{L^2(B_{1/2}(0))}. \qquad (16)\end{aligned}$$

To estimate $\|e^u\|_{L^2(B_{1/2}(0))}$ we apply (10) with $\alpha = 1$, $\beta = 2$ to get

$$\|e^u\|_{L^2(B_{1/2}(0))} \le C\varepsilon^{1/2}. \qquad (17)$$

For $\|u_2\|_{L^2(B_{1/2}(0))}$, first note that

$$\|e^v\|_{L^2(B_{1/2}(0))} \le C\varepsilon^{1/2}.$$

Hence by $L^2$ regularity theory

$$\|u_2\|_{W^{2,2}(B_{1/2}(0))} \le C\varepsilon^{1/2}.$$

By using the Sobolev embedding $W^{2,2} \subset L^{\frac{2n}{n-4}}$ we get

$$\|u_2\|_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \le C\varepsilon^{1/2}. \qquad (18)$$

By interpolation

$$\|u_2\|_{L^2(B_{1/2}(0))} \le \|u_2\|^m_{L^1(B_{1/2}(0))}\|u_2\|^{1-m}_{L^{\frac{2n}{n-4}}(B_{1/2}(0))} \qquad (19)$$

where $m = \frac{4}{n+4} \in (0,1)$. But

$$\|u_2\|_{L^1(B_{1/2}(0))} \le C\lambda\|e^v\|_{L^1(B_{1/2}(0))} \le C\varepsilon, \qquad (20)$$

so (19) combined with (18) and (20) yields

$$\|u_2\|_{L^2(B_{1/2}(0))} \le C\varepsilon^m\varepsilon^{(1-m)/2} = C\varepsilon^{\frac{1+m}{2}}. \qquad (21)$$

Therefore, using (16), (17) and (21) we find

$$\theta^{2-n}\int_{B_\theta(0)\cap[u_2>\varepsilon^\gamma]} e^u \le C\theta^{2-n}\varepsilon^{1+m/2-\gamma}.$$

Combining this and (15) we obtain

$$\theta^{2-n}\int_{B_\theta(0)} e^u \le C\theta^2 e^{\varepsilon^\gamma}\varepsilon + C\theta^{2-n}\varepsilon^{1+m/2-\gamma}.$$

Since $m > 0$ we may choose $0 < \gamma < m/2$. Then fix $\theta > 0$ so that $Ce\theta^2 \le 1/2$ and then choose $\varepsilon_0 > 0$ sufficiently small so that $C\theta^{2-n}\varepsilon_0^{m/2-\gamma} \le 1/2$. It follows that for any $0 < \varepsilon \le \varepsilon_0$

$$\theta^{2-n}\int_{B_\theta(0)} e^u \le \varepsilon.$$

A similar argument yields the corresponding estimate for $e^v$. Rescaling back we obtain (12). $\square$

Applying the previous lemma we can prove

**Lemma 2.9.** *There exists $\varepsilon_1 > 0$ and $\theta > 0$ depending only on $n$ such that for any $0 < \varepsilon \le \varepsilon_1$, if $(u, v)$ is a stable smooth solution of (1), $B_{r_0}(x) \subset \Omega$ and*

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \le \varepsilon$$

*then*

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon$$

*for any $y \in B_{r_0/2}(x)$ and any $0 < r \le r_0/2$.*

*Proof.* By shifting coordinates we can assume that $x = 0$ and by the scaling (13) that $r_0 = 1$. Let $\varepsilon_0$, $\theta$ be the constants of Lemma 2.8. We choose $\varepsilon_1$ so that $2^{n-2}\varepsilon_1 = \varepsilon_0$. Then, for any $y \in B_{1/2}(0)$ and $0 < \varepsilon \le \varepsilon_1$ we have

$$(\frac{1}{2})^{2-n} \int_{B_{1/2}(y)} (e^u + e^v) \le 2^{n-2} \int_{B_1(0)} (e^u + e^v) \le 2^{n-2}\varepsilon \le \varepsilon_0.$$

Applying inductively Lemma 2.8, for any integer $k \ge 1$ we have

$$(\theta^k)^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \le 2^{n-2}\varepsilon.$$

If $0 < r \le 1/2$ is arbitrary we select $k \ge 1$ an integer such that $\theta^{k+1} \le r \le \theta^k$. Then

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le (\theta^{k+1})^{2-n} \int_{B_{\theta^k}(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon.$$

□

*Proof of Lemma 2.7.* The result of Lemma 2.9 holds also for any extremal solution. This can be proved by approximating an extremal solution $(u, v)$ of parameters $(m\lambda^*(m), \lambda^*(m)) \in \partial\mathcal{U}$ by minimal solutions with parameters $(m\lambda, \lambda)$ and $\lambda \uparrow \lambda^*(m)$. In this process, the constants appearing in the estimates remain bounded, see Remark 1.

Let $\varepsilon_1, \theta$ be the constants of Lemma 2.9. We take $0 < \varepsilon_2 < \varepsilon_1$ to be fixed later on. By the change of variables (13) we can assume that $x = 0$ and $r_0 = 1$, so now the hypothesis is

$$\int_{B_1(0)} e^u + e^v \le \varepsilon_2.$$

Then by Lemma 2.9 we have

$$r^{2-n} \int_{B_r(y)} (e^u + e^v) \le 2^{n-2}\theta^{2-n}\varepsilon_2$$

for any $y \in B_{1/2}(0)$ and any $0 < r \le 1/2$. This says that $e^u$, $e^v$ are in the Morrey space $M_{n/2}(B_{1/2}(0))$ and

$$\|e^u\|_{M_{n/2}} + \|e^v\|_{M_{n/2}} \le 2^{n-2}\theta^{2-n}\varepsilon_2. \tag{22}$$

Let $\tilde{u}$, $\tilde{v}$ be the Newtonian potentials of $e^u\chi_{B_{1/2}}(0)$ and $e^v\chi_{B_{1/2}}(0)$ respectively. Then by [10] Lemma 7.20 we have

$$\int_{B_1(0)} e^{\beta|\tilde{u}|} + e^{\beta|\tilde{v}|} \le C_2 \tag{23}$$

for $\beta \le \min(\frac{c_1}{\|e^u\|_{M_{n/2}}}, \frac{c_1}{\|e^v\|_{M_{n/2}}})$ where $c_1, C_2 > 0$ depend only on dimension. By (22), choosing $\varepsilon_2 > 0$ small, we obtain that (23) holds for some $\beta > n/2$. Then $e^u, e^v \in L^\beta(B_{1/4}(0))$ for some $\beta > n/2$. By standard $L^p$ regularity $u, v \in L^\infty(B_{1/8}(0))$. Scaling back we have the conclusion. □

### 2.5. **Proof of Theorem 1.1.**

*Proof.* Let $1 \le \alpha < 5$. We claim that

$$\Sigma \subset \Big\{ x \in \Omega : \limsup_{r \to 0} r^{2\alpha - n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) > 0 \Big\}.$$

Indeed, if $x \in \Omega$ and

$$\lim_{r \to 0} r^{2\alpha - n} \int_{B_r(x) \cap \Omega} (e^{\alpha u} + e^{\alpha v}) = 0$$

then by Hölder's inequality also

$$\lim_{r \to 0} r^{2 - n} \int_{B_r(x) \cap \Omega} (e^{u} + e^{v}) = 0.$$

Therefore for some $r_0 > 0$ so that $B_{r_0}(x) \subset \Omega$ we have

$$r_0^{2-n} \int_{B_{r_0}(x)} (e^u + e^v) \le \varepsilon_2$$

where $\varepsilon_2 > 0$ is the constant from Lemma 2.7. Then by the same lemma $u$, $v$ are bounded in a neighborhood of $x$ and hence $x \notin \Sigma$.

Since $e^{\alpha u} + e^{\alpha v} \in L^1(\Omega)$ by Lemma 2.5, we obtain that $\mathcal{H}^{n-2\alpha}(\Sigma) = 0$, see e.g. [7, Theorem 5.3.4]. Letting $\alpha \uparrow 5$ we deduce that the Hausdorff dimension of $\Sigma$ is less or equal than $n - 10$. □

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